

ON INTEGRAL EQUATIONS RELATED TO WEIGHTED TOEPLITZ OPERATORS

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ABSTRACT. For weighted Toeplitz operators \mathcal{T}_φ^N defined on spaces of holomorphic functions in the unit ball, we derive regularity properties of the solutions f to the integral equation $\mathcal{T}_\varphi^N(f) = h$ in terms of the regularity of the symbol φ and the data h . As an application, we deduce that if $f \not\equiv 0$ is a function in the Hardy space H^1 such that its argument \bar{f}/f is in a Lipschitz space on the unit sphere \mathbb{S} , then f is also in the same Lipschitz space, extending a result of K. Dyakonov to several complex variables.

1. INTRODUCTION

The goal of this paper is to study the regularity of solutions to certain equations related to weighted Toeplitz operators in several complex variables.

We will start by stating some particular cases of the main results in this paper, which involve classical spaces and integral operators and illustrate the object of this paper, although they can be applied in a more general setting.

Let \mathbb{B} denote the open unit ball in \mathbb{C}^n and \mathbb{S} its boundary. In the one variable setting ($n = 1$), \mathbb{B} and \mathbb{S} will also be denoted by \mathbb{D} and \mathbb{T} , respectively. For any $\tau > 0$, $\Lambda_\tau = \Lambda_\tau(\mathbb{S})$ is the classical Lipschitz-Zygmund space on \mathbb{S} .

If $\varphi \in \Lambda_\tau$, we consider the Toeplitz operator $\mathcal{T}_\varphi : H^1 \rightarrow H^1$, defined by $\mathcal{T}_\varphi(f)(z) := \mathcal{P}(\varphi f)(z)$, where \mathcal{P} is the Cauchy projection, given by

$$\mathcal{P}(\psi)(z) := \int_{\mathbb{S}} \frac{\psi(\zeta)}{(1 - \bar{\zeta}z)^n} d\sigma(\zeta) \quad (\psi \in L^1(\mathbb{S})).$$

Here $d\sigma$ denotes the normalized Lebesgue measure on \mathbb{S} . We point out that \mathcal{T}_φ maps H^1 to itself because $\varphi \in \Lambda_\tau$.

Date: September 17, 2010.

2000 Mathematics Subject Classification. 47B35, 46E15, 32A35, 32A37, 30D55.

Key words and phrases. Toeplitz operators, Lipschitz spaces, holomorphic Besov spaces, Hardy spaces.

Partially supported by DGICYT Grant MTM2008-05561-C02-01, DURSI Grant 2009SGR 1303 and Grant MTM2007-30904-E.

For this scale of Lipschitz spaces we prove the following result:

Theorem 1.1. *Let $\tau > 0$ and $\varphi \in \Lambda_\tau$ be a non-vanishing function on \mathbb{S} . If $f \in H^1$ and $\mathcal{T}_\varphi(f) \in \Lambda_\tau$, then $f \in \Lambda_\tau$.*

This result extends [7, Theorem 3.1], which deals with the case $n = 1$ and the regularity of the solutions to the equation $\mathcal{T}_\varphi(f) = 0$.

We remark that if we drop the condition $0 \notin \varphi(\mathbb{S})$, then Theorem 1.1 is not true in general. Indeed, we only need to consider the symbol $\varphi(\zeta) = (1 - \zeta_1)^\tau$ and the function $f(\zeta) = (1 - \zeta_1)^{-\tau}$ with $0 < \tau < n$ (to ensure that $f \in H^1$).

As in the one variable case (see [7]), the above theorem implies some interesting properties of the holomorphic Lipschitz functions. For instance,

Corollary 1.2. *If $f \in H^1$, $\varphi \in \Lambda_\tau$, such that $0 \notin \varphi(\mathbb{S})$ and $\varphi f \in \Lambda_\tau + \ker \mathcal{P} \subset L^1(\mathbb{S})$, then $f \in \Lambda_\tau$.*

In particular, we have:

Corollary 1.3. *If $f \in H^1 \setminus \{0\}$ and its argument function $\varphi = \bar{f}/f$ is in Λ_τ , then $f \in \Lambda_\tau$.*

The preceding corollary is proved in [7] for $n = 1$.

In this paper we prove the above results and extend them to weighted Toeplitz operators associated to more general symbols.

We will denote by $\Gamma_\tau = \Gamma_\tau(\mathbb{S})$, $\tau > 0$, the Lipschitz-Zygmund space on \mathbb{S} with respect to the pseudodistance $d(\zeta, \eta) = |1 - \bar{\zeta}\eta|$ (see Subsection 2.3 for precise definitions). Since $|1 - \bar{\zeta}\eta| \leq |\zeta - \eta|$, it is clear that Γ_τ is a subspace of Λ_τ . For a positive integer k , and real numbers $0 < \tau_0 \leq \tau < k$, $0 < \tau_0 < 1/2$, we will consider spaces $G_{\tau,k}^{\tau_0}(\mathbb{B}) \subset \Lambda_{\tau_0}(\mathbb{B}) \cap \mathcal{C}^k(\mathbb{B})$, whose restrictions to \mathbb{S} contain the space Γ_τ , and also the space Λ_τ , for $\tau > 1/2$. Moreover, they satisfy that their intersection with the space $H = H(\mathbb{B})$ of holomorphic functions on \mathbb{B} , coincides with the Lipschitz-Zygmund space of holomorphic functions on \mathbb{B} , denoted by B_τ^∞ . These holomorphic spaces B_τ^∞ are characterized in terms of the growth of the derivatives (see Subsection 2.3 for a precise definition and their main properties).

For $N > 0$, let $d\nu_N(z) := c_N(1 - |z|^2)^{N-1}d\nu(z)$, where ν is the Lebesgue measure on \mathbb{B} and $c_N = \frac{\Gamma(n+N)}{n!\Gamma(N)}$, so that $\nu_N(\mathbb{B}) = 1$. Let $L_N^1 := L^1(\mathbb{B}, d\nu_N)$ and consider the weighted Bergman projection $\mathcal{P}^N : L_N^1 \rightarrow H$ defined by

$$\mathcal{P}^N(\psi)(z) := \int_{\mathbb{B}} \frac{\psi(w)}{(1 - \bar{w}z)^{n+N}} d\nu_N(w).$$

Let $B_{-N}^1 := L_N^1 \cap H$ and for $\varphi \in L^\infty(\mathbb{B})$, define the weighted Toeplitz operator $\mathcal{T}_\varphi^N : B_{-N}^1 \rightarrow H$ by $\mathcal{T}_\varphi^N(f) := \mathcal{P}^N(\varphi f)$. Since $\lim_{N \searrow 0} \mathcal{P}^N(\psi) = \mathcal{P}(\psi)$ (see [3, § 0.3]), we extend these definitions to $N = 0$, by $\mathcal{P}^0 = \mathcal{P}$ and $\mathcal{T}_\varphi^0 = \mathcal{T}_\varphi$, $\varphi \in L^\infty(\mathbb{S})$. In these cases, the operators are defined on $L^1(\mathbb{S})$ and H^1 , respectively.

The next two theorems are the main results of this paper.

Theorem 1.4. *Let $\varphi \in G_{\tau,k}^{\tau_0}$ such that $0 \notin \varphi(\mathbb{S})$. If $f \in B_{-N}^1$ satisfies $\mathcal{T}_\varphi^N(f) = h \in B_\tau^\infty$, then $f \in B_\tau^\infty$ and $\|f\|_{B_\tau^\infty} \leq C \left(\|f\|_{B_{-N}^1} + \|h\|_{B_\tau^\infty} \right)$, where $C > 0$ is a finite constant only depending on φ , $N > 0$ and n . In particular, $\|f\|_{B_\tau^\infty} \leq C \|f\|_{B_{-N}^1}$, for any $f \in \ker \mathcal{T}_\varphi^N$.*

The corresponding statement for the case $N = 0$ is:

Theorem 1.5. *Let φ be the restriction to \mathbb{S} of a function in $G_{\tau,k}^{\tau_0}$ and let $f \in H^1$. If $0 \notin \varphi(\mathbb{S})$ and $\mathcal{T}_\varphi(f) = h \in B_\tau^\infty$, then $f \in B_\tau^\infty$ and $\|f\|_{B_\tau^\infty} \leq C (\|f\|_{H^1} + \|h\|_{B_\tau^\infty})$, where $C > 0$ is a finite constant only depending on φ and n . In particular, $\|f\|_{B_\tau^\infty} \leq C \|f\|_{H^1}$, for any $f \in \ker \mathcal{T}_\varphi$.*

The preceding theorem was proved in [7, Theorem 3.1] for $n = 1$, $\varphi \in \Lambda_\tau$ and $h = 0$.

Note that the inequalities in the above theorems are in fact equivalences due to the continuity of both the Toeplitz operator and the embeddings $B_\tau^\infty \subset H^1 \subset B_{-N}^1$.

For $\tau > 1/2$, the restriction to \mathbb{S} of $G_{\tau,k}^{\tau_0}$ contains the space Λ_τ , hence Theorem 1.5 includes the result of Theorem 1.1 for these cases. However, the same techniques used to prove the above theorems allow us to extend this result to the whole scale of spaces Λ_τ .

Corollary 1.6. *If either $f \in B_{-N}^1$, for $N > 0$, or $f \in H^1$ and $N = 0$, $\varphi \in G_{\tau,k}^{\tau_0}$, $0 \notin \varphi(\mathbb{S})$ and $\varphi f \in G_{\tau,k}^{\tau_0} + \ker \mathcal{P}^N$, then $f \in B_\tau^\infty$.*

In particular if $g \in H^1$, then $\overline{g - g(0)} \in \ker \mathcal{P}$, and therefore we have:

Corollary 1.7. *If $f, g \in H^1$ satisfy $\varphi = \bar{g}/f \in \Gamma_\tau$ and $0 \notin \varphi(\mathbb{S})$, then $f, g \in B_\tau^\infty$.*

The preceding result generalizes Corollary 1.3, and extends [7, Corollary 3.2] to dimension $n > 1$.

The paper is organized as follows. In Section 2 we state some properties of spaces considered in this paper and we also recall some integral representation formulas used in the proof of the main theorems.

In Section 3 we state our main technical theorem (Theorem 3.1) from which we deduce Theorems 1.4 and 1.5 and its corollaries. We

also construct some counterexamples. Finally, Theorem 3.1 is proved in Section 4.

2. PRELIMINARIES

2.1. Notations. Throughout the paper, the letter C will denote a positive constant, which may vary from place to place. The notation $f(z) \lesssim g(z)$ means that there exists $C > 0$, which does not depend on z , f and g , such that $f(z) \leq Cg(z)$. We write $f(z) \approx g(z)$ when $f(z) \lesssim g(z)$ and $g(z) \lesssim f(z)$.

Let $\partial_j := \frac{\partial}{\partial z_j}$, for $j = 1, \dots, n$. For any multiindex α , i.e. $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, where \mathbb{N} is the set of non-negative integers, let $|\alpha| := \sum_{j=1}^n \alpha_j$ and $\partial_\alpha := \frac{\partial^{|\alpha|}}{\partial z^\alpha} = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. We write $|\partial^k \varphi| := \sum_{|\alpha|=k} |\partial_\alpha \varphi|$ and $|d^k \varphi| := \sum_{|\alpha|+|\beta|=k} |\partial_\alpha \bar{\partial}_\beta \varphi|$. When $n > 1$, we also consider the complex tangential differential operators $\mathcal{D}_{i,j} := \bar{z}_i \partial_j - \bar{z}_j \partial_i$ and $|\partial_T \varphi| := \sum_{1 \leq i < j \leq n} |\mathcal{D}_{i,j} \varphi|$. For $n = 1$, we write $|\partial_T \varphi| := 0$.

We also introduce the following two functions which will be used in the definition of the spaces $G_{\tau,k}^{\tau_0}$.

For $\varphi \in \mathcal{C}^1(\mathbb{B})$, let

$$(2.1) \quad \tilde{\varphi}(z) := (1 - |z|^2) |\bar{\partial} \varphi(z)| + (1 - |z|^2)^{1/2} |\bar{\partial}_T \varphi(z)|.$$

For $t \in \mathbb{R}$, let ω_t be the function on \mathbb{B} defined by

$$(2.2) \quad \omega_t(z) := \begin{cases} (1 - |z|^2)^{\min(t,0)}, & \text{if } t \neq 0, \\ \log \frac{e}{1-|z|^2}, & \text{if } t = 0. \end{cases}$$

2.2. Holomorphic Besov spaces. Let $1 \leq p < \infty$, $k \in \mathbb{N}$ and $\delta > 0$. The weighted Sobolev space $L_{k,\delta}^p$ is the completion of the space $\mathcal{C}^\infty(\bar{\mathbb{B}})$, endowed with the norm

$$\|\psi\|_{L_{k,\delta}^p} := \left\{ \sum_{j=0}^k \int_{\mathbb{B}} |d^j \psi(z)|^p (1 - |z|^2)^{\delta p - 1} d\nu(z) \right\}^{1/p}.$$

When $k = 0$, we will just write $L_\delta^p = L_{0,\delta}^p$. We extend this definition to the case $p = \infty$, so that $L_{k,\delta}^\infty$ is the subspace of functions ψ in the Sobolev space $L_{k,\delta+1}^1$ satisfying

$$\|\psi\|_{L_{k,\delta}^\infty} := \sum_{j=0}^k \sup_{z \in \mathbb{B}} |d^j \psi(z)| (1 - |z|^2)^\delta < \infty.$$

If $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, the holomorphic Besov space B_s^p is defined to be $B_s^p := H \cap L_{k,k-s}^p$, for some $k \in \mathbb{N}$, $k > s$. It is well known that

$\|\cdot\|_{L_{m,m-s}^p}$ and $\|\cdot\|_{L_{k,k-s}^p}$ are equivalent norms on B_s^p , for any $m, k \in \mathbb{N}$, $m, k > s$.

Note that if $s = 0$, then B_0^∞ is the Bloch space and if $s > 0$ then B_s^∞ coincides with the space of holomorphic functions on \mathbb{B} whose boundary values are in the corresponding Lipschitz-Zygmund space Λ_s (see the next subsection for the precise definitions of these last two spaces).

Proposition 2.1 ([3, Theorems 5.13,14]). *Let $1 \leq p \leq q \leq \infty$ and let $s, t \in \mathbb{R}$. Then:*

- (i) *If $s > t$, then $B_s^p \subset B_t^p$.*
- (ii) *For any $\varepsilon > 0$, $B_0^1 \subset H^1 \subset B_{-\varepsilon}^1$.*
- (iii) *If $s - n/p = t - n/q$, then $B_{s+n/p}^1 \subset B_s^p \subset B_t^q \subset B_{s-n/p}^\infty$.*

2.3. The space $G_{\tau,k}^{\tau_0}$. In this section we define the spaces $G_{\tau,k}^{\tau_0}$ and we state some of their main properties.

Definition 2.2. *Let $0 < \tau_0 \leq \tau$, $\tau_0 < 1/2$, and let $k > \tau$ be an integer. The space $G_{\tau,k}^{\tau_0}$ consists of all functions $\varphi \in \mathcal{C}^k(\mathbb{B}) \cap \mathcal{C}(\bar{\mathbb{B}})$ satisfying*

$$\|\varphi\|_{G_{\tau,k}^{\tau_0}} := \sum_{j=0}^k \sup_{z \in \mathbb{B}} \frac{|\partial^j \varphi(z)|}{\omega_{\tau-j}(z)} + \sup_{z \in \mathbb{B}} \frac{\tilde{\varphi}(z)}{(1 - |z|^2)^{\tau_0}} < \infty,$$

Note that $H \cap G_{\tau,k}^{\tau_0} = B_\tau^\infty$, and $G_{\tau,k}^{\tau_0} \cdot L_\delta^p \subset L_\delta^p$, since $G_{\tau,k}^{\tau_0} \subset L^\infty(\mathbb{B})$.

The following embedding is a consequence of the definition of $G_{\tau,k}^{\tau_0}$ and the fact that $(1 - |z|^2)^s \leq (1 - |z|^2)^t$ and $\omega_s(z) \lesssim \omega_t(z)$, if $s > t$.

Lemma 2.3. $G_{\tau,k}^{\tau_0} \subset G_{\vartheta,m}^{\vartheta_0}$, *provided that $\vartheta_0 \leq \tau_0$, $\vartheta \leq \tau$ and $m \leq k$.*

In order to obtain multiplicative properties of the spaces $G_{\tau,k}^{\tau_0}$, we first state some properties of the function ω_t .

Lemma 2.4. *Let $a, b \in \mathbb{R}$. Then*

$$(1 - |z|^2)^a \omega_b(z) \lesssim (1 - |z|^2)^c,$$

for every $c \in \mathbb{R}$ such that $c < a$ and $c \leq a + b$.

Proof. Just note that

$$(1 - |z|^2)^a \omega_b(z) = \begin{cases} (1 - |z|^2)^{\min(a+b,a)}, & \text{if } b \neq 0 \\ (1 - |z|^2)^a \log \frac{e}{1-|z|^2}, & \text{if } b = 0 \end{cases} \lesssim (1 - |z|^2)^c,$$

for every $c \in \mathbb{R}$ such that $c < a$ and $c \leq a + b$. ■

Lemma 2.5. *Let $\vartheta, \tau > 0$, $k \in \mathbb{R}$ and $m \in \mathbb{Z}$ such that $m \geq 0$. Then*

$$S_{m,k}^{\vartheta,\tau} := \sum_{i=0}^m \omega_{\vartheta-i} \omega_{\tau+i-k} \lesssim \omega_{\vartheta-m} + \omega_{\tau-k}.$$

Proof. We estimate the different products $\omega_{\vartheta-i} \omega_{\tau+i-k}$ as follows:

- If $i > k - \tau$, then $\omega_{\vartheta-i} \omega_{\tau+i-k} = \omega_{\vartheta-i} \lesssim \omega_{\vartheta-m}$, since $\vartheta - m \leq \vartheta - i$.
- If $i < \vartheta$, then $\omega_{\vartheta-i} \omega_{\tau+i-k} = \omega_{\tau+i-k} \lesssim \omega_{\tau-k}$, since $\tau - k \leq \tau + i - k$.
- If $i > \vartheta$ and $i \leq k - \tau$, then

$$\omega_{\vartheta-i}(z) \omega_{\tau+i-k}(z) = (1 - |z|)^{\vartheta-i} \omega_{\tau+i-k}(z) \lesssim (1 - |z|)^{\tau-k} = \omega_{\tau-k}(z),$$

by Lemma 2.4, since $\tau - k < \tau - k + \vartheta = (\vartheta - i) + (\tau + i - k)$ and $\tau - k \leq -i < -\vartheta < 0$.

- If $i = \vartheta$ and $i < k - \tau$, then

$$\omega_{\vartheta-i}(z) \omega_{\tau+i-k}(z) = (1 - |z|)^{\tau+i-k} \omega_0(z) \lesssim (1 - |z|)^{\tau-k} = \omega_{\tau-k}(z),$$

by Lemma 2.4, since $\tau - k < \tau - k + \vartheta = \tau + i - k < 0$.

- If $\vartheta = i = k - \tau$, then

$$\omega_{\vartheta-i}(z) \omega_{\tau+i-k}(z) = \omega_0(z)^2 \lesssim (1 - |z|)^{\tau-k} = \omega_{\tau-k}(z),$$

since $\tau - k = -\vartheta < 0$. ■

Proposition 2.6.

$$\|\varphi\psi\|_{G_{\vartheta,m}^{\vartheta_0}} \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|\psi\|_{G_{\vartheta,m}^{\vartheta_0}} \quad (\varphi \in G_{\tau,k}^{\tau_0}, \psi \in G_{\vartheta,m}^{\tau_0}),$$

provided that $\vartheta_0 \leq \tau_0$, $\vartheta \leq \tau$ and $m \leq k$.

Proof. If $\alpha \in \mathbb{N}^n$, $|\alpha| = l \leq m$, then

$$\begin{aligned} |\partial_\alpha(\varphi\psi)| &\lesssim \sum_{\beta+\gamma=\alpha} |\partial_\beta\varphi| |\partial_\gamma\psi| \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|\psi\|_{G_{\vartheta,m}^{\vartheta_0}} S_{l,l}^{\vartheta,\tau} \\ &\lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|\psi\|_{G_{\vartheta,m}^{\vartheta_0}} \omega_{\vartheta-l}, \end{aligned}$$

since, by Lemma 2.5, $S_{l,l}^{\vartheta,\tau} \lesssim \omega_{\vartheta-l} + \omega_{\tau-l} \lesssim \omega_{\vartheta-l}$.

On the other hand,

$$\widetilde{\varphi\psi}(z) \leq |\varphi(z)| \widetilde{\psi}(z) + \widetilde{\varphi}(z) |\psi(z)| \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|\psi\|_{G_{\vartheta,m}^{\vartheta_0}} (1 - |z|^2)^{\vartheta_0},$$

and the proof is complete. ■

Our next goal is to show the connection between the spaces $G_{\tau,k}^{\tau_0}$ and both the non-isotropic Lipschitz-Zygmund spaces Γ_τ and the classical Lipschitz-Zygmund spaces Λ_τ .

If $0 < \tau < 1$, the classical Lipschitz-Zygmund space on \mathbb{S} , $\Lambda_\tau = \Lambda_\tau(\mathbb{S})$, with respect to the Euclidean metric consists of all the functions $\varphi \in \mathcal{C}(\mathbb{S})$ such that

$$\|\varphi\|_{\Lambda_\tau} := \|\varphi\|_\infty + \sup_{\substack{\zeta, \eta \in \mathbb{S} \\ \zeta \neq \eta}} \frac{|\varphi(\zeta) - \varphi(\eta)|}{|\zeta - \eta|^\tau} < \infty.$$

If k is a positive integer and $k < \tau < k + 1$, then $\Lambda_\tau = \Lambda_\tau(\mathbb{S})$ consists of all the functions $\varphi \in \mathcal{C}^k(\mathbb{S})$ such that

$$\|\varphi\|_{\Lambda_\tau} := \|\varphi\|_{\mathcal{C}^k} + \sum_{|\alpha|+|\beta|=k} \|\partial_\alpha \bar{\partial}_\beta \varphi\|_{\Lambda_{\tau-k}(\mathbb{S})} < \infty.$$

When τ is a positive integer, Λ_τ is defined analogously by using second order differences. The spaces $\Lambda_\tau(\mathbb{B})$ are defined in a similar way.

The main properties of the spaces Λ_τ can be found, for instance, in the expository paper [8].

It is well known (see [8, § 15]) that a continuous function φ is in Λ_τ if and only if, for some (any) integer $k > \tau$, its harmonic extension Φ on \mathbb{B} satisfies

$$(2.3) \quad \sup_{z \in \mathbb{B}} (1 - |z|^2)^{k-\tau} |d^k \Phi(z)| < \infty.$$

We recall that if (2.3) holds for some function $\varphi \in \mathcal{C}^k(\mathbb{B})$, then $\varphi \in \Lambda_\tau(\mathbb{B})$ (see [8, Theorem 15.7]).

This fact and the estimate $|d\varphi(z)| \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} (1 - |z|^2)^{\tau_0-1}$ give

Proposition 2.7. $G_{\tau,k}^{\tau_0} \subset \Lambda_{\tau_0}(\mathbb{B})$.

We also consider the Lipschitz-Zygmund space on \mathbb{S} with respect to the pseudodistance $d(\zeta, \eta) = |1 - \bar{\zeta}\eta|$, which is denoted by $\Gamma_\tau(\mathbb{S})$. If $0 < \tau < 1/2$, this space is defined just as Λ_τ but replacing the Euclidean distance $|\zeta - \eta|$ by $d(\zeta, \eta)$. For values $\tau \geq 1/2$ the definition is given in terms of Lipschitz conditions of certain complex tangential derivatives (see [4, pp. 670-1] and the references therein for the precise definitions and main properties).

We recall that if $f \in H(\mathbb{B})$ has boundary values f^* , then f^* is in Λ_τ , if and only if $f^* \in \Gamma_\tau$ (see [13] or [12, §6.4] or [10, §8.8] and the references therein. See also [4, pp. 670-1]). If $0 < \tau < n$, the functions in Γ_τ can be described in terms of their invariant harmonic extensions. In this case, we have that φ is in Γ_τ if and only if, for some (any) integer $k > \tau$, its invariant harmonic extension Φ on \mathbb{B} satisfies (2.3). This characterization fails to be true when $\tau \geq n$ (see [9, Chapter 6] for more details). Similarly to what happens in the holomorphic case, the complex tangential derivatives of the functions in the space Γ_τ are more regular, in the sense that $\mathcal{D}_{ij}\varphi \in \Gamma_{\tau-1/2}$ for $i, j = 1, \dots, n$.

The next results relate the spaces Λ_τ and Γ_τ to $G_{\tau,k}^{\tau_0}$.

Proposition 2.8.

a) If $n = 1$ then the harmonic extension of a function in Λ_τ belongs to any space $G_{\tau,k}^{\tau_0}$.

b) If $n > 1$ and $\tau > 1/2$ then every $\varphi \in \Lambda_\tau$ is the restriction of a function $\Phi \in G_{\tau,k}^{\tau_0}$. Namely, for any integer $k > \tau$, the harmonic extension Φ of φ satisfies that:

- $\Phi \in G_{\tau,k}^{\tau-1/2}$, when $1/2 < \tau < 1$.
- $\Phi \in G_{\tau,k}^{\tau_0}$, for any $0 < \tau_0 < 1/2$, when $\tau \geq 1$.

Corollary 2.9. *If either $n = 1$ or $n > 1$ and $\tau > 1/2$, then every $\varphi \in \Lambda_\tau$ is the restriction of a function $\Phi \in G_{\tau,k}^{\tau_0}$, for some $0 < \tau_0$ and for any integer k .*

Proposition 2.10. *If $n > 1$, $0 < \tau < n$ and $\varphi \in \Gamma_\tau$, then, for any integer $k > \tau$, its invariant harmonic extension Φ satisfies that:*

- $\Phi \in G_{\tau,k}^\tau$, when $0 < \tau < 1/2$.
- $\Phi \in G_{1,k}^{\tau_0}$, for any $0 < \tau_0 < 1/2$, when $\tau \geq 1/2$.

Corollary 2.11. *Every $\varphi \in \Gamma_\tau$, $\tau > 0$, is the restriction of a function $\Phi \in G_{\tau,k}^{\tau_0}$, for some $0 < \tau_0$ and for any integer k .*

2.4. Representation formulas and estimates. In this subsection we recall some well-known results on the integral representation formulas obtained in [6].

We begin by introducing the following nonnegative integral kernels and their corresponding integral operators.

Definition 2.12. *Let $N, M, L \in \mathbb{R}$ such that $N > 0$ and $L < n$. Then*

$$\mathcal{K}_{M,L}^N(w, z) := \frac{(1 - |w|^2)^{N-1}}{|1 - \bar{w}z|^M D(w, z)^L}, \quad (z, w \in \bar{\mathbb{B}}, z \neq w),$$

where $D(w, z) := |1 - \bar{w}z|^2 - (1 - |w|^2)(1 - |z|^2)$. The associated integral operator is also denoted by $\mathcal{K}_{M,L}^N$:

$$\mathcal{K}_{M,L}^N(\psi)(z) := \int_{\bar{\mathbb{B}}} \mathcal{K}_{M,L}^N(w, z) \psi(w) d\nu(w).$$

Note that $D(w, z) = |(w - z)\bar{z}|^2 + (1 - |z|^2)|w - z|^2$, so, for every $z \in \bar{\mathbb{B}}$ such that $1 - |z|^2 \geq \delta > 0$, we have that

$$(2.4) \quad \mathcal{K}_{M,L}^N(w, z) \simeq \begin{cases} |w - z|^{-2L}, & \text{if } |w - z| < (1 - |z|)/2, \\ (1 - |w|^2)^{N-1}, & \text{if } |w - z| \geq (1 - |z|)/2. \end{cases}$$

Theorem 2.13 ([6]). *Let $N > 0$. Then every function $\psi \in \mathcal{C}^1(\bar{\mathbb{B}})$ decomposes as*

$$(2.5) \quad \psi = \mathcal{P}^N(\psi) + \mathcal{K}^N(\bar{\partial}\psi),$$

where

$$\mathcal{K}^N(\bar{\partial}\psi)(z) := \int_{\bar{\mathbb{B}}} \mathcal{K}^N(w, z) \wedge \bar{\partial}\psi(w)$$

and $\mathcal{K}^N(w, z)$ is an $(n, n-1)$ -form (on w) of class \mathcal{C}^∞ on $\mathbb{B} \times \mathbb{B}$ outside its diagonal.

In particular if ψ is holomorphic on \mathbb{B} then $\psi = \mathcal{P}^N(\psi)$.

Moreover, $\mathcal{K}^N(w, z)$ satisfies the estimate

$$(2.6) \quad |\mathcal{K}^N(w, z) \wedge \bar{\partial}\psi(w)| \lesssim K_{N-n+1, n-1/2}^N(w, z) \tilde{\psi}(w),$$

for any $\psi \in \mathcal{C}^1(\mathbb{B})$, where $\tilde{\psi}$ is defined as in (2.1).

Then, it is clear that,

$$(2.7) \quad |\mathcal{P}^N(\psi)| \lesssim \mathcal{K}_{n+N, 0}^N(|\psi|) \quad \text{and} \quad |\mathcal{K}^N(\bar{\partial}\psi)| \lesssim \mathcal{K}_{N-n+1, n-1/2}^N(\tilde{\psi}).$$

Remark 2.14. The above representation formula will be applied in a more general setting to functions $\psi = \varphi f$ where $\varphi \in G_{\tau, k}^{\tau_0}$ and either $f \in B_{-N}^1$, for $N > 0$, or $f \in H^1$, for $N = 0$. The validity of the formula for this class of functions is obtained by applying the dominated convergence theorem and Theorem 2.13 to the functions $\psi_r(z) = \psi(rz)$.

Lemma 2.15 ([6, Lemma I.1]).

$$\int_{\mathbb{B}} \mathcal{K}_{M, L}^N(w, z) d\nu(w) \lesssim \omega_t(z),$$

where $t := n + N - M - 2L$ is the so-called **type** of the kernel $\mathcal{K}_{M, L}^N$.

Observe that from the above estimate we deduce that if $\mathcal{K}_{M, L}^N$ is a kernel of type 0, $0 < \delta < N$ and $\psi(z) = (1 - |z|^2)^{-\delta}$, then $\mathcal{K}_{M, L}^N(\psi) \lesssim \psi$. As a consequence of that result and Schur's lemma we have:

Lemma 2.16. If $\mathcal{K}_{M, L}^N$ is a kernel of type 0 and $0 < \delta < N$, then $\mathcal{K}_{M, L}^N$ maps boundedly L_δ^p to itself.

By applying Hölder's inequality we deduce the following pointwise estimate of the operators $\mathcal{K}_{M, L}^N$, which will be often used in the forthcoming sections.

Lemma 2.17. Let $N \geq 0$, $\tau > 0$, $p \geq 1$ and $0 < \varepsilon < N + \tau$. Then $\left(\mathcal{K}_{N-n+1, n-1/2}^{N+\tau}(|\psi|)\right)^p \lesssim \mathcal{K}_{Np-n+1, n-1/2}^{(N+\tau-\varepsilon)p}(|\psi|^p)$.

In the next lemma we state some differentiation formulas for both operators \mathcal{P}^N and \mathcal{K}^N .

Lemma 2.18. Let $N \geq 0$, $\alpha \in \mathbb{N}^n$ and $k = |\alpha|$.

- (i) If $\psi \in \mathcal{C}^k(\mathbb{B})$, then $\partial_\alpha \mathcal{P}^N(\psi) = \mathcal{P}^{N+k}(\partial_\alpha \psi)$.
- (ii) If $\psi \in \mathcal{C}^{k+1}(\mathbb{B})$, then $\partial_\alpha \mathcal{K}^N(\bar{\partial}\psi) = \mathcal{K}^{N+k}(\bar{\partial}\partial_\alpha \psi)$.

Proof. These results are well known (see, for instance, [5, § 5]). For the sake of completeness, we give a brief sketch of the proof. For $N > 0$, (i) follows from the equation $\frac{\partial}{\partial z_j} \mathcal{P}^N(w, z) = \frac{\partial}{\partial w_j} \mathcal{P}^{N+1}(w, z)$ and integration by parts, while (ii) is just a direct consequence of (2.5) and (i).

The case $N = 0$ is deduced from the corresponding formulas for $N > 0$ by taking $N \searrow 0$. \blacksquare

Remark 2.19. *The above differentiation formulas will be applied to functions $\psi = \varphi f$ where $\varphi \in G_{\tau,k}^{\tau_0}$ and $f \in B_s^\infty$, $s > 0$. The validity of the formulas in this more general setting can be shown by applying Lemma 2.18 to $\psi_r(z) = \psi(rz)$ and the dominated convergence theorem.*

Now we state some regularity properties related to the integral operator \mathcal{P}^N .

Proposition 2.20.

- (i) *If $0 < \delta < N$ and $1 \leq p < \infty$, then \mathcal{P}^N maps continuously L_δ^p in $B_{-\delta}^p$.*
- (ii) *If $N \geq 0$, then \mathcal{P}^N maps continuously $G_{\tau,k}^{\tau_0}$ in B_τ^∞ .*
- (iii) *If $N = 0$, then \mathcal{P} maps continuously Λ_τ in B_τ^∞ .*

Proof. The proof of (i) can be found in [14, Theorem 2.10]. The proof of (ii) reduces to show that every $\psi \in G_{\tau,k}^{\tau_0}$ satisfies

$$|\partial^k \mathcal{P}^N(\psi)(z)| = |\mathcal{P}^{N+k}(\partial^k \psi)(z)| \lesssim K_{n+N+k,0}^{N+\tau}(1)(z) \lesssim (1 - |z|^2)^{\tau-k},$$

which follows from Lemmas 2.18 and 2.15. Assertion (iii) can be found in [12, § 6.4]. \blacksquare

Proposition 2.21. *If $\varphi \in G_{\tau,k}^{\tau_0}$, then \mathcal{T}_φ^N maps boundedly B_{-N}^1 to itself, for $N > 0$, and H^1 to itself, for $N = 0$.*

Proof. The second assertion is a consequence of $G_{\tau,k}^{\tau_0} \subset \Lambda_{\tau_0}$ and [12, Theorem 6.5.4], so let us prove the first one. Assume $N > 0$. By Proposition 2.7, $\varphi \in \Lambda_{\tau_0}(\mathbb{B})$, so $|\varphi(w) - \varphi(z)| \lesssim |w - z|^{\tau_0} \lesssim |1 - \bar{w}z|^{\tau_0/2}$. Then, since $\mathcal{T}_\varphi^N(f)(z) = \mathcal{T}_{\varphi-\varphi(z)}^N(f)(z) + \varphi(z)f(z)$, we have that

$$|\mathcal{T}_\varphi^N(f)| \lesssim \mathcal{K}_{n+N-\tau_0/2,0}^N(|f|) + |f|.$$

By Fubini's Theorem and Lemma 2.15, $\|\mathcal{K}_{n+N-\tau_0/2,0}^N(|f|)\|_{L_N^1} \lesssim \|f\|_{L_N^1}$. Therefore $\|\mathcal{T}_\varphi^N(f)\|_{L_N^1} \lesssim \|f\|_{L_N^1}$, and the proof is complete. \blacksquare

3. TOEPLITZ OPERATORS WITH SYMBOLS IN $G_{\tau,k}^{\tau_0}$

In this section we state a general theorem from which we will deduce the results stated in the introduction. The proof of this general theorem will be postponed to the next section.

Observe that if the functions $\varphi \in G_{\tau,k}^{\tau_0}$ and $f \in B_{-N}^1$, $N \geq 0$, satisfy the equation $\mathcal{T}_\varphi^N(f) = h \in B_\tau^\infty$, then, taking into account Remark 2.14, formula (2.5) gives that

$$(3.8) \quad \varphi f = \mathcal{K}^N(f\bar{\partial}\varphi) + h.$$

Note that, by (2.6),

$$|\mathcal{K}^N(f\bar{\partial}\varphi)| \lesssim \mathcal{K}_{N-n+1,n-1/2}^N(|f|\bar{\varphi}) \lesssim \mathcal{K}_{N-n+1,n-1/2}^{N+\tau_0}(|f|)$$

and therefore by (2.4), $\mathcal{K}^N(f\bar{\partial}\varphi)$ is pointwise defined even if $f \in B_{-N_0}^1$, for some $N < N_0 < N + \tau_0$. This fact and the inclusion $H^1 \subset B_{-N_0}^1$ for any $N_0 > 0$, allow us to unify the proofs of Theorems 1.4 and 1.5, using the following result:

Theorem 3.1. *Let $N \geq 0$ and let $\varphi \in G_{\tau,k}^{\tau_0}$ be such that $0 \notin \varphi(\mathbb{S})$. If $0 < N_0 < N + \tau_0$, $f \in B_{-N_0}^1$ and $h \in B_\tau^\infty$ satisfy (3.8), then $f \in B_\tau^\infty$ and $\|f\|_{B_\tau^\infty} \lesssim \|f\|_{B_{-N_0}^1} + \|h\|_{B_\tau^\infty}$.*

Now we easily deduce Theorems 1.4 and 1.5 all at once:

Theorem 3.2. *Let $\varphi \in G_{\tau,k}^{\tau_0}$ be such that $0 \notin \varphi(\mathbb{S})$.*

- (i) *If $f \in B_{-N}^1$ and $\mathcal{T}_\varphi^N(f) \in B_\tau^\infty$, for some $N > 0$, then $f \in B_\tau^\infty$ and $\|f\|_{B_\tau^\infty} \lesssim \|f\|_{B_{-N}^1} + \|h\|_{B_\tau^\infty}$.*
- (ii) *If $f \in H^1$ and $\mathcal{T}_\varphi(f) \in B_\tau^\infty$, then $f \in B_\tau^\infty$ and $\|f\|_{B_\tau^\infty} \lesssim \|f\|_{H^1} + \|h\|_{B_\tau^\infty}$.*

Proof. As we pointed out at the beginning of the section, if $\mathcal{T}_\varphi^N(f) = h \in B_\tau^\infty$, $N \geq 0$, then φ and f satisfy (3.8). Therefore (i) directly follows from Theorem 3.1 (case $N > 0$). By Proposition 2.1, $H^1 \subset B_{-t}^1$, for every $t > 0$, and, in particular, $H^1 \subset B_{-N_0}^1$, for every $0 < N_0 < \tau_0$, so (ii) also follows from Theorem 3.1 (case $N = 0$). \blacksquare

As an immediate consequence of Theorem 3.2 we obtain the following corollaries.

Corollary 3.3. *Let $\tau > 0$ and assume that φ satisfy that $0 \notin \varphi(\mathbb{S})$, and one of the following conditions:*

- (i) $n = 1$ and $\varphi \in \Lambda_\tau$.
- (ii) $n > 1$, $\tau > \frac{1}{2}$ and $\varphi \in \Lambda_\tau$.
- (iii) $n > 1$, $\tau \leq \frac{1}{2}$ and $\varphi \in \Gamma_\tau$.

If $f \in H^1$ and $\mathcal{T}_\varphi(f) \in B_\tau^\infty$, then $f \in B_\tau^\infty$.

Proof. This is a consequence of Theorem 3.2 and Corollaries 2.9 and 2.11. \blacksquare

Corollary 3.4. Let $\varphi \in G_{\tau,k}^{\tau_0}$ be such that $0 \notin \varphi(\mathbb{S})$.

- (i) If $N > 0$ and $f \in B_{-N}^1$ satisfies that $\varphi f \in G_{\tau,k}^{\tau_0} + \ker \mathcal{P}^N$, then $f \in B_\tau^\infty$.
- (ii) If $f \in H^1$ satisfies that $\varphi f \in G_{\tau,k}^{\tau_0} + \ker \mathcal{P}$, then $f \in B_\tau^\infty$.

Proof. This is a consequence of Theorem 3.2 and Proposition 2.20(ii). \blacksquare

Corollary 3.5. If $\varphi \in G_{\tau,k}^{\tau_0}$, then $\ker(\mathcal{T}_\varphi^N - \lambda \mathcal{I}) \subset B_\tau^\infty$, for any $\lambda \in \mathbb{C} \setminus \varphi(\mathbb{S})$ and $N \geq 0$. In particular, $\ker \mathcal{T}_\varphi^N \subset B_\tau^\infty$, whenever $0 \notin \varphi(\mathbb{S})$.

Proof. Since $\mathcal{T}_\varphi^N - \lambda \mathcal{I} = \mathcal{T}_{\varphi-\lambda}^N$, it directly follows from Theorem 3.2. \blacksquare

Remark 3.6. If the condition $0 \notin \varphi(\mathbb{S})$ is omitted, then $\ker \mathcal{T}_\varphi^N$ is not necessarily contained in B_τ^∞ . For $n > 1$, this result follows by taking $\varphi(z) = \bar{z}_1$, and observing that $\ker \mathcal{T}_\varphi^N$ contains any function in B_{-N}^1 , if $N > 0$, (H^1 , if $N = 0$), which does not depend on the first variable. For $n = 1$ we may consider the symbol $\varphi(z) = \bar{z}^{m+1}(1-z)^{m+\alpha}$ and the function $f(z) = (1-z)^{-\alpha}$, where $0 < \alpha < 1$ and $m \in \mathbb{N}$ such that $m + \alpha \geq \tau$, which satisfy $\varphi \in G_{\tau,k}^{\tau_0}$ and $f \in \ker \mathcal{T}_\varphi^N \setminus B_\tau^\infty$.

Now we extend Corollary 3.3.

Theorem 3.7. Let $\tau > 0$ and let $\varphi \in \Lambda_\tau$ be a non-vanishing function on \mathbb{S} . If $f \in H^1$ and $\mathcal{T}_\varphi(f) \in B_\tau^\infty$, then $f \in B_\tau^\infty$.

Proof. If $\tau > 1/2$, the result is just a consequence of Corollary 2.9 and part (ii) of Theorem 3.2.

Now assume that $\tau \leq 1/2$. Since $|w - z|^2 \leq 2|1 - \bar{w}z|$, we have that $\Lambda_\tau \subset \Gamma_{\tau/2}$. And then Corollary 2.11 and part (ii) of Theorem 3.2 show that $f \in B_{\tau/2}^\infty$. Thus $|\partial f(z)| \lesssim (1 - |z|^2)^{\tau/2-1}$, but we want to prove that $|\partial f(z)| \lesssim (1 - |z|^2)^{\tau-1}$, or equivalently $|\Phi(z)\partial f(z)| \lesssim (1 - |z|^2)^{\tau-1}$, Φ being the harmonic extension of φ to \mathbb{B} . (Recall that, since $0 \notin \varphi(\mathbb{S})$, there is $0 < r < 1$ so that $|\Phi(z)| \simeq 1$ for $r \leq |z| \leq 1$.)

In order to show the estimate note that $|d\Phi(z)| \lesssim (1 - |z|^2)^{\tau-1}$, which implies that $|\Phi(z) - \Phi(w)| \lesssim |z - w|^\tau \lesssim |1 - \bar{w}z|^{\tau/2}$, for $z, w \in \mathbb{B}$. On the other hand, since $f \in B_{\tau/2}^\infty$, $\partial_j f \in B_{-1}^1$ so $\partial_j f = \mathcal{P}^1(\partial_j f)$ and therefore

$$\Phi(z)\partial_j f(z) = \mathcal{P}^1((\Phi(z) - \Phi)\partial_j f)(z) + \mathcal{P}^1(\partial_j(\Phi f))(z) - \mathcal{P}^1(f\partial_j\Phi)(z).$$

By Lemma 2.18, $\mathcal{P}^1(\partial_j(\Phi f)) = \partial_j \mathcal{T}_\varphi f$. Hence

$$|\Phi(z)\partial_j f(z)| \lesssim \mathcal{K}_{n+1-\tau/2,0}^{\tau/2}(1)(z) + (1-|z|^2)^{\tau-1} + \mathcal{K}_{n+1,0}^\tau(1)(z),$$

and then Lemma 2.15 shows that $|\Phi(z)\partial_j f(z)| \lesssim (1-|z|^2)^{\tau-1}$. \blacksquare

Since \mathcal{P} maps Λ_τ to B_τ^∞ , we deduce

Corollary 3.8. *If $f \in H^1$ and $\varphi \in \Lambda_\tau$ satisfy $0 \notin \varphi(\mathbb{S})$ and $\varphi f \in \Lambda_\tau + \ker \mathcal{P}$, then $f \in B_\tau^\infty$.*

Now we obtain Corollary 1.3:

Corollary 3.9. *If $f, g \in H^1 \setminus \{0\}$ satisfy $\varphi = \bar{g}/f \in \Lambda_\tau$ and $0 \notin \varphi(\mathbb{S})$, then $f, g \in B_\tau^\infty$. In particular, if $f \in H^1 \setminus \{0\}$ and its argument function \bar{f}/f is in Λ_τ , then $f \in B_\tau^\infty$.*

Proof. Since $\mathcal{T}_\varphi(f) = \mathcal{P}(\bar{g}) = \overline{g(0)} \in B_\tau^\infty$, Theorem 3.7 shows that $f \in B_\tau^\infty$. Therefore $\bar{g} = f\varphi \in \Lambda_\tau$ and hence $g \in B_\tau^\infty$. \blacksquare

4. PROOF OF THEOREM 3.1

This section is devoted to the proof of Theorem 3.1. It is split into three steps composed of several lemmas that will give successive improvements on the regularity of the solutions to the equation $\mathcal{T}_\varphi^N(f) = h$. First we will show that any solution f to (3.8) which is in $B_{-N_0}^1$ is in fact in any B_{-t}^1 , $t > 0$. Then we will obtain that the solution is in B_{-t}^∞ for any $t > 0$, and finally we will deduce that it is in B_τ^∞ .

Throughout this section we will assume that φ and h satisfy the hypotheses of Theorem 3.1. Since $|\varphi(\zeta)| \geq \rho > 0$ on \mathbb{S} , we can choose r_0 such that $|\varphi(z)| \geq \rho/2 > 0$ on the corona $C = \{z \in \mathbb{B} : r_0 \leq |z| \leq 1\}$. Let χ be a real \mathcal{C}^∞ -function on \mathbb{C}^n supported on the corona $C_0 = \{z \in \mathbb{B} : r_0 \leq |z| \leq 1 + r_0\}$, such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on a neighborhood of \mathbb{S} . Then (3.8) shows that

$$(4.9) \quad f = \frac{\chi}{\varphi} \mathcal{K}^N(f \bar{\partial} \varphi) + \frac{\chi}{\varphi} h + (1 - \chi)f.$$

The function $(1 - \chi)f$ is a \mathcal{C}^∞ function with compact support on \mathbb{B} . It is easy to prove that $\frac{\chi}{\varphi} \in G_{\tau,k}^{\tau_0}$, and so $\frac{\chi}{\varphi} h \in G_{\tau,k}^{\tau_0}$, by Proposition 2.6. Therefore $(1 - \chi)f + \frac{\chi}{\varphi} h \in G_{\tau,k}^{\tau_0}$ and

$$(4.10) \quad \|(1 - \chi)f + \frac{\chi}{\varphi} h\|_{G_{\tau,k}^{\tau_0}} \leq C_\varphi \left(\|f\|_{B_{-N_0}^1} + \|h\|_{B_\tau^\infty} \right)$$

Hence, in order to prove that $f \in B_\tau^\infty$, we have just to show that $\frac{\chi}{\varphi} \mathcal{K}^N(f\bar{\partial}\varphi) \in G_{\tau,k}^{\tau_0}$.

STEP 1. The first couple of lemmas will show that $f \in B_{-t}^1$, for any $t > 0$.

Lemma 4.1. *Let $f \in B_{-s}^1$, for some $0 < s < N + \tau_0$, and assume it satisfies (3.8).*

- (i) *If $s \leq \tau_0$ then $f \in B_{-t}^1$, for every $t > 0$.*
- (ii) *If $s > \tau_0$ then $f \in B_{-(s-\tau_0)}^1$.*

Proof. First note that (2.7) shows that

$$|\mathcal{K}^N(f\bar{\partial}\varphi)| = |\mathcal{K}^N(\bar{\partial}(f\varphi))| \lesssim \mathcal{K}_{N-n+1, n-\frac{1}{2}}^N(|f|\tilde{\varphi}),$$

and so

$$(4.11) \quad |\mathcal{K}^N(f\bar{\partial}\varphi)| \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \mathcal{K}_{N-n+1, n-1/2}^{N+\tau_0}(|f|).$$

By integrating and using Fubini's Theorem, for any $t > 0$ we have that

$$\|\mathcal{K}^N(f\bar{\partial}\varphi)\|_{L_t^1} \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \int_{\mathbb{B}} |f(w)| g_t(w) d\nu(w),$$

where

$$g_t(w) = (1 - |w|^2)^{N+\tau_0-1} \int_{\mathbb{B}} \mathcal{K}_{N-n+1, n-1/2}^t(z, w) d\nu(z).$$

Now Lemmas 2.15 and 2.4 show that

$$(4.12) \quad g_t(w) \lesssim (1 - |w|^2)^{N+\tau_0-1} \omega_{t-N}(w) \lesssim (1 - |w|^2)^{s-1},$$

provided that $s \leq t + \tau_0$. (recall that $s < N + \tau_0$). Therefore, if $s \leq t + \tau_0$,

$$(4.13) \quad \|\mathcal{K}^N(f\bar{\partial}\varphi)\|_{L_t^1} \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_{L_s^1},$$

so, by (4.9) and (4.10), $f \in B_{-t}^1$. We conclude that:

(i) If $s \leq \tau_0$ then $s \leq t + \tau_0$ and (4.12) holds for every $t > 0$, and hence $f \in B_{-t}^1$, for every $t > 0$.

(ii) If $s > \tau_0$ then (4.12) holds for $t = s - \tau_0$, and consequently $f \in B_{-(s-\tau_0)}^1$. \blacksquare

Lemma 4.2. *If $f \in B_{-N_0}^1$ satisfies (3.8) then $f \in B_{-t}^1$, for every $t > 0$.*

Proof. For $N_0 \leq \tau_0$ the result follows directly from Lemma 4.1 (i). So assume that $N_0 > \tau_0$. Let k be the greatest positive integer such that $k\tau_0 < N_0$. Then $k\tau_0 < N_0 \leq (k+1)\tau_0$. Now, since $f \in B_{-N_0}^1$,

Lemma 4.1(ii) implies that $f \in B_{-(N_0-\tau_0)}^1$, so $f \in B_{-(N_0-2\tau_0)}^1, \dots$, so $f \in B_{-(N_0-k\tau_0)}^1$. But $0 < N_0 - k\tau_0 \leq \tau_0$ and therefore Lemma 4.1(i) shows that $f \in B_{-t}^1$, for every $t > 0$. \blacksquare

Remark 4.3. Observe that the above arguments, (4.9) and (4.13) give in particular the estimate $\|f\|_{B_{-t}^1} \lesssim \|f\|_{B_{-N_0}^1} + \|h\|_{B_{\tilde{r}}^\infty}$.

STEP 2. The next couple of lemmas will show that the function f is in B_{-t}^∞ , for any $t > 0$. We follow the ideas in [7].

Lemma 4.4. Let $f \in B_{-s}^p$, for some $1 \leq p < \infty$ and for every $s > 0$. If f satisfies (3.8) then $f \in B_{-s}^q$, for every $s > 0$ and for every q such that $p < q < \infty$ and $\frac{1}{p} - \frac{\tau_0}{n} < \frac{1}{q}$.

Proof. If $-t < -s < 0$, then the space $B_{-s}^p \subset B_{-t}^p$. Consequently, we only have to prove the lemma, for s sufficiently small. Let $p < q < \infty$ and $0 < \varepsilon < N + \tau_0$. Assume f satisfies (3.8). Then, as we have shown in the proof of Lemma 4.1, (4.11) holds, and so Lemma 2.17 gives

$$|\mathcal{K}^N(f\bar{\partial}\varphi)|^q \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}}^q \mathcal{K}_{Nq-n+1, n-\frac{1}{2}}^{(N+\tau_0-\varepsilon)q}(|f|^q).$$

By Proposition 2.1(iii), $B_{-s}^p \subset B_{-s-n/p}^\infty$ and $|f(w)| \lesssim \|f\|_{B_{-s}^p} (1 - |w|^2)^{-s-\frac{n}{p}}$, which implies that

$$|f(w)|^q = |f(w)|^{q-p} |f(w)|^p \lesssim \|f\|_{B_{-s}^p}^{q-p} (1 - |w|^2)^{(p-q)(s+\frac{n}{p})} |f(w)|^p,$$

and, by integrating, we get

$$\mathcal{K}_{Nq-n+1, n-\frac{1}{2}}^{(N+\tau_0-\varepsilon)q}(|f|^q) \lesssim \|f\|_{B_{-s}^p}^{q-p} \mathcal{K}_{M,L}^{N(\varepsilon,s)}(|f|^p),$$

where $N(\varepsilon, s) = (N + \tau_0 - \varepsilon)q + (p - q) \left(s + \frac{n}{p}\right) = sp + (N - s)q + nq \left(\frac{\tau_0 - \varepsilon}{n} - \frac{1}{p} + \frac{1}{q}\right)$, $M = Nq - n + 1$ and $L = n - 1/2$.

Therefore

$$\|\mathcal{K}^N(f\bar{\partial}\varphi)\|_{L_s^q} \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_{B_{-s}^p}^{1-\frac{p}{q}} I_{\varepsilon,s}, \quad \text{where } I_{\varepsilon,s} = \|\mathcal{K}_{M,L}^{N(\varepsilon,s)}(|f|^p)\|_{L_{sq}^1}.$$

Thus we only have to prove that $I_{\varepsilon,s}^q \lesssim \|f\|_{B_{-s}^p}^p$, for $\varepsilon, s > 0$ small enough and $\frac{1}{p} - \frac{\tau_0}{n} < \frac{1}{q}$, because then the previous estimate shows that $\|\mathcal{K}^N(f\bar{\partial}\varphi)\|_{L_s^q} \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_{B_{-s}^p}$ and hence, by (4.9) and (4.10), we conclude that $f \in B_{-s}^q$. In order to estimate $I_{\varepsilon,s}^q$, first apply Fubini's Theorem to get

$$I_{\varepsilon,s}^q = \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^{N(\varepsilon,s)-1} \left(\int_{\mathbb{B}} \mathcal{K}_{M,L}^{sq}(z, w) d\nu(z) \right) d\nu(w),$$

and since $n + sq - M - 2L = (s - N)q$, then apply Lemma 2.15 to obtain

$$I_{\varepsilon, s}^q \lesssim \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^{N(\varepsilon, s) - 1} \omega_{(s-N)q}(w) d\nu(w).$$

Now we consider two cases:

Case $N = 0$. Then $(s - N)q = sq > 0$ so

$$I_{\varepsilon, s}^q \lesssim \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^{N(\varepsilon, s) - 1} d\nu(w) \leq \|f\|_{B_{-s}^p}^p,$$

provided that $N(\varepsilon, s) > sp$, which holds for $\varepsilon, s > 0$ small enough and $\frac{1}{p} - \frac{1}{q} < \frac{\tau_0}{n}$, since

$$\lim_{\varepsilon, s \searrow 0} (N(\varepsilon, s) - sp) = nq \left\{ \frac{\tau_0}{n} - \left(\frac{1}{p} - \frac{1}{q} \right) \right\} > 0.$$

Case $N > 0$. Let $0 < s < N$. Then $(s - N)q < 0$ and so

$$I_{\varepsilon, s}^q \lesssim \int_{\mathbb{B}} |f(w)|^p (1 - |w|^2)^{N(\varepsilon, s) + (s-N)q - 1} d\nu(w) \leq \|f\|_{B_{-s}^p}^p,$$

provided that $N(\varepsilon, s) + (s - N)q > sp$, which holds for $\varepsilon > 0$ small enough and $\frac{1}{p} - \frac{1}{q} < \frac{\tau_0}{n}$, since

$$N(\varepsilon, s) + (s - N)q - sp = nq \left\{ \frac{\tau_0 - \varepsilon}{n} - \left(\frac{1}{p} - \frac{1}{q} \right) \right\} > 0.$$

And the proof is complete. ■

Lemma 4.5. *Let $f \in B_{-s}^1$, for every $s > 0$. If f satisfies (3.8) then $f \in B_{-t}^\infty$, for every $t > 0$.*

Proof. Let k be the greatest positive integer such that $k \frac{\tau_0}{2n} < 1$. Then $k \frac{\tau_0}{2n} < 1 \leq (k + 1) \frac{\tau_0}{2n}$. Let

$$p_j = \frac{1}{1 - j \frac{\tau_0}{2n}} \quad (j = 0, \dots, k).$$

Then $p_j \geq 1$, for $j = 0, \dots, k$, and $\frac{1}{p_j} = \frac{1}{p_{j-1}} - \frac{\tau_0}{2n} > \frac{1}{p_{j-1}} - \frac{\tau_0}{n}$, for $j = 1, \dots, k$. Now, since $f \in B_{-s}^{p_0}$, for every $s > 0$, and f satisfies (3.8), Lemma 4.4 shows that $f \in B_{-s}^{p_1}$ so $f \in B_{-s}^{p_2}, \dots$, so $f \in B_{-s}^{p_k}$, for every $s > 0$. But $\frac{1}{p_k} - \frac{\tau_0}{2n} = 1 - (k + 1) \frac{\tau_0}{2n} \leq 0$ and therefore Lemma 4.4 once again shows that $f \in B_{-s}^q$, for every $q > p_k$ and every $s > 0$. Since $B_{-s}^q \subset B_{-s-\frac{n}{q}}^\infty$, by Proposition 2.1(iii), we conclude that $f \in B_{-t}^\infty$, for every $t > 0$. ■

Remark 4.6. *Observe that the above arguments and (4.9) give the estimate $\|f\|_{B_{-t}^\infty} \lesssim \|f\|_{B_{-t}^1} + \|h\|_{B_{-t}^\infty}$.*

STEP 3. In what follows we will finally deduce that $f \in B_\tau^\infty$.

Lemma 4.7. *Let $f \in B_{-t}^\infty$, for every $t > 0$. If f satisfies (3.8) then $f \in H^\infty$.*

Proof. Since f satisfies (3.8), (4.11) holds, as we have shown in the proof of Lemma 4.1. But

$$\mathcal{K}_{N-n+1, n-\frac{1}{2}}^{N+\tau_0}(|f|) \lesssim \|f\|_{B_{-t}^\infty} \mathcal{K}_{N-n+1, n-\frac{1}{2}}^{N+\tau_0-t}(1)$$

and, by Lemma 2.15, $\mathcal{K}_{N-n+1, n-\frac{1}{2}}^{N+\tau_0-t}(1) \lesssim \omega_{\tau_0-t} \lesssim 1$, for any $0 < t < \tau_0$. Therefore $\|\mathcal{K}^N(f\bar{\partial}\varphi)\|_\infty \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_{B_{-t}^\infty}$, and, by (3.8) and (4.10) $f \in H^\infty$. \blacksquare

In order to prove that $f \in B_\tau^\infty$, we will use the following formula.

Lemma 4.8. *Let $N \geq 0$, $\varphi \in G_{\tau,k}^{\tau_0}$ and $f \in H^\infty$. Then*

$$(4.14) \quad \varphi \partial_\alpha f = \partial_\alpha \mathcal{P}^N(\varphi f) - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|<k}} c_{\alpha,\beta} \mathcal{P}^{N+k}(\partial_\gamma \varphi \partial_\beta f) + \mathcal{K}^{N+k}(\bar{\partial}\varphi \partial_\alpha f),$$

for every $\alpha \in \mathbb{N}^n$, where $k = |\alpha|$ and $c_{\alpha,\beta} = \alpha! / (\beta! \gamma!)$.

Proof. First assume that $\varphi \in \mathcal{C}^k(\bar{\mathbb{B}})$ and $f \in H(\bar{\mathbb{B}})$. By Theorem 2.13 we have that $\varphi \partial_\alpha f = \mathcal{P}^{N+k}(\varphi \partial_\alpha f) + \mathcal{K}^{N+k}(\bar{\partial}\varphi \partial_\alpha f)$. Moreover,

$$\varphi \partial_\alpha f = \partial_\alpha(\varphi f) - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|<k}} c_{\alpha,\beta} \partial_\gamma \varphi \partial_\beta f,$$

and Lemma 2.18 shows that $\mathcal{P}^{N+k}(\partial_\alpha(\varphi f)) = \partial_\alpha \mathcal{P}^N(\varphi f)$. Hence we obtain (4.14).

By a standard approximation argument (based on the dominated convergence theorem), we deduce the general case from the regular case just proved above. \blacksquare

Lemma 4.9.

- (i) *If $f \in H^\infty$ satisfies (3.8), then for every $0 < t \leq \tau_0$, $f \in B_t^\infty$.*
- (ii) *Let $f \in B_s^\infty$, for some $0 < s < \tau$. If f satisfies (3.8), then for every $0 < t \leq \min(\tau, s + \tau_0)$, $f \in B_t^\infty$.*

Proof. Let f and t be as in either (i) or (ii), and assume f satisfies (3.8). Let $k \in \mathbb{Z}$ and $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k > \tau$. We want to prove that $|\partial_\alpha f(z)| \lesssim (1 - |z|^2)^{t-k}$. Since

$$(4.15) \quad |\partial_\alpha f| \leq \|\chi/\varphi\|_\infty |\varphi \partial_\alpha f| + (1 - \chi) |\partial_\alpha f|,$$

we only need to estimate $|\varphi \partial_\alpha f|$.

Lemma 4.8, (3.8) and (2.7) show that

$$(4.16) \quad |\varphi \partial_\alpha f| \lesssim |\partial_\alpha h| + \mathcal{K}_{n+N+k,0}^{N+k}(F_\alpha) + \mathcal{K}_{N+k-n+1,n-\frac{1}{2}}^{N+k}(|\partial_\alpha f| \tilde{\varphi}),$$

where $F_\alpha = \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|<k}} |\partial_\beta f| |\partial_\gamma \varphi|$. Note that we only need to prove that

$$|\partial_\alpha f(w)| \tilde{\varphi}(w) \lesssim (1 - |w|^2)^{t-k} \quad \text{and} \quad F_\alpha(w) \lesssim (1 - |w|^2)^{t-k},$$

because then (4.16) and Lemma 2.15 show that

$$\begin{aligned} |(\varphi \partial_\alpha f)(z)| &\lesssim |\partial_\alpha h(z)| + \mathcal{K}_{N+k-n+1,n-\frac{1}{2}}^{N+t}(1)(z) + \mathcal{K}_{n+N+k,0}^{N+t}(1)(z) \\ &\lesssim \omega_{t-k}(z) + \omega_{\tau-k}(z) = (1 - |z|^2)^{t-k}, \end{aligned}$$

and therefore, by (4.15), we conclude that $|\partial_\alpha f(z)| \lesssim (1 - |z|^2)^{t-k}$.

(i) Let $f \in H^\infty$. Then $|\partial_\beta f(w)| \lesssim \|f\|_\infty (1 - |w|^2)^{-|\beta|}$, for every multi-index β . So

$$|\partial_\alpha f(w)| \tilde{\varphi}(w) \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_\infty (1 - |w|^2)^{\tau_0-k} \lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_\infty (1 - |w|^2)^{t-k},$$

for every $t \in \mathbb{R}$ such that $t \leq \tau_0$. Next Lemma 2.4 shows that

$$\begin{aligned} F_\alpha(w) &\lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_\infty \sum_{i=0}^{k-1} (1 - |w|^2)^{-i} \omega_{\tau-k+i}(z) \\ &\lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_\infty (1 - |w|^2)^{t-k}, \end{aligned}$$

for every $t \in \mathbb{R}$ such that $t < 1$ and $t \leq \tau$.

(ii) Let $f \in B_s^\infty$, for some $0 < s < \tau$. Then $|\partial_\beta f(z)| \lesssim \|f\|_{B_s^\infty} \omega_{s-|\beta|}(z)$, for every multiindex β , so Lemma 2.4 gives that

$$\begin{aligned} |\partial_\alpha f(w)| \tilde{\varphi}(w) &\lesssim \|f\|_{B_s^\infty} \|\varphi\|_{G_{\tau,k}^{\tau_0}} (1 - |w|^2)^{s+\tau_0-k} \\ &\leq \|f\|_{B_s^\infty} \|\varphi\|_{G_{\tau,k}^{\tau_0}} (1 - |w|^2)^{t-k}, \end{aligned}$$

for every $t \in \mathbb{R}$ such that $t \leq s + \tau_0 < s + 1$, and Lemma 2.5 shows that

$$\begin{aligned} F_\alpha(w) &\lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_{B_s^\infty} S_{k-1,k}^{s,\tau}(w) \\ &\lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_{B_s^\infty} \{\omega_{s+1-k}(w) + (1 - |w|^2)^{\tau-k}\} \\ &\lesssim \|\varphi\|_{G_{\tau,k}^{\tau_0}} \|f\|_{B_s^\infty} (1 - |w|^2)^{t-k}, \end{aligned}$$

for every $t \in \mathbb{R}$ such that $t \leq s + 1$ and $t \leq \tau$. ■

Lemma 4.10. *If $f \in H^\infty$ satisfies (3.8) then $f \in B_\tau^\infty$.*

Proof. Let $f \in H^\infty$. If $\tau = \tau_0$ there is nothing to prove by Lemma 4.9 (i). So assume that $\tau_0 < \tau$, and let $k \geq 1$ be the greatest integer such that $k\tau_0 < \tau$. Since $f \in H^\infty$, Lemma 4.9 (i) shows that $f \in B_{\tau_0}^\infty$, and then Lemma 4.9 (ii) implies that $f \in B_{2\tau_0}^\infty$, so $f \in B_{3\tau_0}^\infty, \dots$, so $f \in B_{k\tau_0}^\infty$, and hence $f \in B_\tau^\infty$. ■

Remark 4.11. *Observe that the above arguments and (4.9) give the estimate $\|f\|_{B_\tau^\infty} \lesssim \|f\|_{B_{-\tau}^\infty} + \|h\|_{B_\tau^\infty}$.*

Remark 4.12. *The remarks 4.3, 4.6 and 4.11 show that*

$$\|f\|_{B_\tau^\infty} \lesssim \|f\|_{B_{-N_0}^1} + \|h\|_{B_\tau^\infty}.$$

Note that the opposite estimate is always fulfilled. This follows from the continuous embedding $B_\tau^\infty \subset B_{-N_0}^1$, and the estimate $\|h\|_{B_\tau^\infty} \leq \|\mathcal{T}_\varphi^N\| \|f\|_{B_\tau^\infty}$.

Therefore

$$\|f\|_{B_\tau^\infty} \approx \|f\|_{B_{-N}^1} + \|h\|_{B_\tau^\infty}.$$

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